A subsolution for TU games(Developments of probability models on optimization problems)

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1. Introduction and Preliminaries

In this paper we consider a subset of the imputation set for cooperative TU games, and examine its properties.

An n-person cooperative game with side payments (abbreviated as a *game*) is an ordered pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of *players* and v, called the *characteristic function*, is a real-valued function on the power set of N, satisfying $v(\emptyset) = 0$. For simplicity we express a game (N, v) as v. A subset of N is called a *coalition*. For any set Z, $|Z|$ denotes the cardinality of Z. For $S \subseteq N$ and $x \in \mathbf{R}^N$, we define $x(S) = \sum_{i\in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). A *pre-imputation* for a game v is a vector $x \in \mathbf{R}^N$ that satisfies

$$x(N) = v(N). \tag{1.1}$$

We denote by $\mathcal{PI} \equiv \mathcal{PI}(v)$ the set of all pre-imputations for a game v. A pre-imputation $x \in \mathcal{PI}$ is said to be *individually rational* if[1]

$$x_i \geq v(i), \quad \forall i \in N. \tag{1.2}$$

An individually rational pre-imputation is called an *imputation*. We denote by $\mathcal{I} \equiv \mathcal{I}(v)$ the set of all imputations for a game v, which we call the imputation set. A pre-imputation $x \in \mathcal{PI}$ is said to be *reasonable* if

$$x_i \leq u_i, \quad \forall i \in N, \tag{1.3}$$

where $u_i \equiv u_i(v) \equiv \max_{i\in S}\{v(S) - v(S\backslash\{i\})\}$ for all $i \in N$. We denote by $\mathcal{R} \equiv \mathcal{R}(v)$ the set of all reasonable pre-imputations for a game v. For $x, y \in \mathcal{I}$ and for a coalition $S \subset N$, we say that x dominates y via S, denoted by $x \succ_S y$, if

$$\begin{cases} \text{(i) } x_i > y_i, & \forall i \in S, \\ \text{(ii) } x(S) \leq v(S). \end{cases} \tag{1.4}$$

For $x, y \in \mathcal{I}$, we say that x dominates y, denoted by $x \succ y$, if there is an S such that x

[1]For simplicity, we write $v(\{i\}), v(\{i, j\})$ as $v(i), v(ij)$.

dominates y via S. For $\mathcal{X} \subseteq \mathcal{I}$, we denote by Dom $\mathcal{X}$ the set of all imputations dominated by some element of $\mathcal{X}$. A set of imputations $\mathcal{X} \subseteq \mathcal{I}$ is called a *stable set* if it satisfies

$$\begin{cases} \text{(i) } \mathcal{X} \cap \mathrm{Dom}\mathcal{X} = \emptyset & \text{(internal stability)} , \\ \\ \text{(ii) } \mathcal{X} \cup \mathrm{Dom}\mathcal{X} = \mathcal{I} & \text{(external stability)} . \end{cases} \tag{1.5}$$

The *core* of a game v, denoted by $\mathcal{C} \equiv \mathcal{C}(v)$, is defined by

$$\mathcal{C} = \mathcal{I} \setminus \mathrm{Dom}\mathcal{I}. \tag{1.6}$$

2. A Subsolution

In this section we define a subset $\mathcal{Q}$ of the imputation set and examine properties of it. We assume that for every game in this section the imputation set is not empty, $\mathcal{I}(v) \neq \emptyset$, that is,

$$v(N) \geq \sum_{i \in N} v(i). \tag{2.1}$$

Definition. A set $\mathcal{Q} \equiv \mathcal{Q}(v) \subseteq \mathcal{I}$ is defined by

$$\mathcal{Q} \equiv \{x \in \mathcal{I} : \forall y \in \mathcal{I} \text{ s.t. } y \succ x, \quad \exists z \in \mathcal{I} \text{ s.t. } z \succ y \text{ and } z \not\succ x.\}.$$

Remark. Let

$$\mathcal{Q}' \equiv \{x \in \mathcal{I} : \forall y \in \mathcal{I} \text{ s.t. } y \succ x, \quad \exists z \in \mathcal{I} \text{ s.t. } z \succ y.\}.$$

If $\mathcal{C} = \emptyset$ then $\mathcal{I} =$ Dom $\mathcal{I}$. And so $\mathcal{Q}' = \mathcal{I}$.

Hereafter we fix a game (N, v).

Proposition 2.1. For a game v, let $\mathcal{X}$ be a stable set. Then $\mathcal{X} \subseteq \mathcal{Q}$.
Proof: Let $x \in \mathcal{X}$ and suppose $y \succ x$ where $y \in \mathcal{I}$. By the internal stability, we have $y \notin \mathcal{X}$, and so by the external stability, there exists $z \in \mathcal{X}$ such that $z \succ y$. By the internal stability, $z \not\succ x$. Hence $x \in \mathcal{Q}$. □

Proposition 2.2. For a game v, it holds $\mathcal{Q} \subseteq \mathcal{R}$.
Proof: Let $x \in \mathcal{Q}$ and assume $x \notin \mathcal{R}$. There exists $i \in N$ such that $x_i > u_i$. This implies $x_i > v(N) - v(N \setminus \{i\})$, which implies $x(N \setminus \{i\}) < v(N \setminus \{i\})$. Hence we can take $y \in \mathcal{I}$ such that $y \succ x$ via $N \setminus \{i\}$ and $y_i \geq u_i$. Since $x \in \mathcal{Q}$, there exists $z \in \mathcal{I}$ such that $z \succ y$ via a coalition S and $z \not\succ x$. If $i \notin S$ then $z \succ x$ via S, which is a contradiction. If $z(S \setminus i) \leq v(S \setminus i)$ then $z \succ x$ via $S \setminus \{i\}$, which is a contradiction. So we must have $i \in S$ and $z(S \setminus \{i\}) > v(S \setminus \{i\})$. Then $y_i < z_i = z(S) - z(S \setminus \{i\}) < v(S) - v(S \setminus \{i\}) \leq u_i$, which is a contradiction. □

From this proposition we see that if $v(S \cup \{i\}) = v(S) + v(i)$ for all $S : i \notin S$ and $x \in \mathcal{Q}$ then it must hold $x_i = v(i)$ since $u_i = v(i)$.

Proposition 2.3. For a game v, it holds $\mathcal{C} \subseteq \mathcal{Q} \subseteq \mathcal{I} \setminus \mathrm{Dom}\mathcal{C}$.
Proof: By Proposition 2.1, we have $\mathcal{C} \subseteq \mathcal{Q}$. If $x \in \mathrm{Dom}\mathcal{C}$, then there exists $y \in \mathcal{C}$ such that $y \succ x$ and $y \notin \mathrm{Dom}\mathcal{I}$. Hence $x \in \mathcal{Q}$. □

Proposition 2.4. For a game v, the core $\mathcal{C}$ is a stable set if and only if $\mathcal{C} = \mathcal{Q}$.
Proof: Assume that $\mathcal{C}$ is a stable set. By Proposition 2.3, we have $\mathcal{C} \subseteq \mathcal{Q}$. Let $x \in \mathcal{Q} \setminus \mathcal{C}$. Since $\mathcal{C}$ is a stable set, by the external stability there exists $y \in \mathcal{C}$ such that $y \succ x$. But there exists no imputation which dominates y because y is in the core. This is a contradiction. Hence $\mathcal{Q} \setminus \mathcal{C} = \emptyset$.

Conversely assume $\mathcal{C} = \mathcal{Q}$. Since $\mathcal{C} \subseteq \mathcal{X}$ for any stable set $\mathcal{X}$, we have $\mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}$ from Proposition 2.1. Hence $\mathcal{C}$ is a unique stable set. □

Proposition 2.5. Suppose (N, v) is symmetric, that is, v depends on only the number of members in a coalition. For every $S \subseteq N$, let $v(s) = v(S)$ where $s = |S|$. Assume $v(1) = 0$. Then

$$x^* \equiv (\frac{v(n)}{n}, \ldots, \frac{v(n)}{n}) \in \mathcal{Q}.$$

Proof: Suppose $y \succ x^*$ via S and $y \not\succ x^*$ via every T such that $T \subset S, T \neq S$. Then

$$y_j > \frac{v(n)}{n}, \forall j \in S, \quad y(S) \leq v(S), \text{ and } y(T) > v(T), \forall T \subset S, T \neq S.$$

Then

$$v(n) = y(N) = y(N \setminus S) + y(S) > y(N \setminus S) + \frac{|S|}{n} v(n).$$

This implies $\frac{n-|S|}{n} v(n) > y(N \setminus S)$, and so there exists $i \in N \setminus S$ such that $y_i < \frac{v(n)}{n}$. For some $j_0 \in S$, let $S^0 = (S \setminus \{j_0\}) \cup \{i\}$. Define $z \in \mathcal{PI}$ by

$$z_j = \begin{cases} y_j + \epsilon, & j \in S^0 = (S \setminus \{j_0\}) \cup \{i\}; \\ \frac{v(n)}{n} - \delta, & j \in N \setminus S^0. \end{cases}$$

Then $y(S^0) < y(S) \leq v(S) = v(S^0)$, and so for sufficiently small $\epsilon > 0$, we have

$$z(S^0) = y(S^0) + \epsilon |S^0| \leq v(S^0), \quad z_j > y_j, \forall j \in S^0.$$

Hence $z \succ y$ via S^0, and $z \not\succ x^*$ since $z_i = y_i + \epsilon \le \frac{v(n)}{n}$ and $z(T) = y(T) + \epsilon|T| > y(T) > v(T)$ for every $T \subset S^0 \setminus \{j_0\}$. It remains to see that it is possible to find $\epsilon > 0$ and $\delta > 0$ such that $z \in \mathcal{I}$ and $z(S^0) \le v(S^0)$. $z(N) = v(n)$ if and only if

$$y(S^0 \setminus \{i\}) - \frac{|S^0| - 1}{n} v(n) + \epsilon|S^0| = \frac{v(n)}{n} - y_i + \delta(n - |S^0|). \tag{2.2}$$

$0 < \delta \le \frac{v(n)}{n}$ if and only if

$$\frac{v(n)}{n} - \frac{y(S^0)}{|S^0|} < \epsilon \le \frac{v(n) - y(S^0)}{|S^0|}. \tag{2.3}$$

$z(S^0) \le v(S^0)$ if and only if

$$\epsilon \le \frac{v(S^0) - y(S^0)}{|S^0|}. \tag{2.4}$$

Since $x^*(S) < y(S) \le v(S) = v(S^0)$, we have $\frac{v(n)}{n} < \frac{v(S^0)}{|S^0|}$. Hence there exist ϵ and δ which satisfy (2.2) – (2.4). $\square$

Proposition 2.5 implies that $\mathcal{Q} \neq \emptyset$ when v is symmetric.

Definition. (Roth (1976)) A set $\mathcal{Y} \subseteq \mathcal{I}$ is called a *subsolution* if

$$\begin{cases} (i)\mathcal{Y} \subseteq \mathcal{I} \setminus \mathrm{Dom}\mathcal{Y}, & \text{(internal stability)} \\ \\ (ii)\mathcal{Y} = \mathcal{I} \setminus \mathrm{Dom}(\mathcal{I} \setminus \mathrm{Dom}\mathcal{Y}). & \end{cases}$$

Proposition 2.6. Let $\mathcal{Y}$ be a subsolution. Then $\mathcal{Y} \subseteq \mathcal{Q}$.
Proof: Let $\mathcal{Y}$ be a subsolution and suppose $x \in \mathcal{Y}$. For any $y \in \mathcal{I}$ such that $y \succ x$, it holds $y \notin \mathcal{Y}$ since $\mathcal{Y}$ is internally stable. So $y \notin \mathcal{I} \setminus \mathrm{Dom}(\mathcal{I} \setminus \mathrm{Dom}\mathcal{Y})$ by the definition of subsolution. Hence $y \in \mathrm{Dom}(\mathcal{I} \setminus \mathrm{Dom}\mathcal{Y})$. This implies that there exists $z \in \mathcal{I} \setminus \mathrm{Dom}\mathcal{Y}$ such that $z \succ y$. Since $x \notin \mathrm{Dom}(\mathcal{I} \setminus \mathrm{Dom}\mathcal{Y})$, it holds that $z \not\succ x$. Hence $x \in \mathcal{Q}$. $\square$

The next example says that the set $\mathcal{Q}$ is different from the union of all stable sets. A remaining problem is whether the set $\mathcal{Q}$ coincides or not with the union of all stable sets when there exists a stable set.

Example 2.1. The 10-Person Game (Lucas (1969)). Let us consider the 10-person game:

$$\begin{aligned} &v(N) = 5, v(13579) = 4, v(3579) = v(1579) = v(1379) = 3, \\ &v(1479) = v(3679) = v(2579) = 2, v(357) = v(157) = v(137) = 2, \\ &v(359) = v(159) = v(139) = 2, v(12) = v(34) = v(56) = v(78) = v(90) = 1, \\ &v(i) = 0 \quad \forall i \in N \end{aligned}$$

and, for other S, $v(S) = 0$. Let

$$\mathcal{B} = \{x \in \mathcal{I} : x(12) = x(34) = x(56) = x(78) = x(90) = 1, \quad x_i \geq 0, \forall i \in N\}.$$

It is easy to check that the core $\mathcal{C}$ of this game is :

$$\mathcal{C} = \{x \in \mathcal{B} : x(13579) \geq 4\}.$$

Define the following subsets of $\mathcal{B}$:

$$\begin{aligned}
\mathcal{E}_1 &= \{x \in \mathcal{B} : x_3 = x_5 = 1, x_1 < 1, x(79) < 1\},\\
\mathcal{E}_3 &= \{x \in \mathcal{B} : x_5 = x_1 = 1, x_3 < 1, x(79) < 1\},\\
\mathcal{E}_5 &= \{x \in \mathcal{B} : x_1 = x_3 = 1, x_5 < 1, x(79) < 1\},\\
\mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3,
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{35} &= \{x \in \mathcal{B} : x(35) = 1, x_1 < 1, x(79) \geq 1\} \setminus \mathcal{C},\\
\mathcal{F}_{51} &= \{x \in \mathcal{B} : x(15) = 1, x_3 < 1, x(79) \geq 1\} \setminus \mathcal{C},\\
\mathcal{F}_{13} &= \{x \in \mathcal{B} : x(13) = 1, x_5 < 1, x(79) \geq 1\} \setminus \mathcal{C},\\
\mathcal{F}_7 &= \{x \in \mathcal{B} : x_7 = 1, x_9 < 1, x(359) \geq 2, x(159) \geq 2, x(139) \geq 2\} \setminus \mathcal{C},\\
\mathcal{F}_9 &= \{x \in \mathcal{B} : x_9 = 1, x_7 < 1, x(357) \geq 2, x(157) \geq 2, x(137) \geq 2\} \setminus \mathcal{C},\\
\mathcal{F}_{79} &= \{x \in \mathcal{B} : x_7 = x_9 = 1\} \setminus \mathcal{C},\\
\mathcal{F}_{135} &= \{x \in \mathcal{B} : x_1 = x_3 = x_5 = 1\} \setminus \mathcal{C},\\
\mathcal{F} &= \mathcal{F}_{13} \cup \mathcal{F}_{35} \cup \mathcal{F}_{51} \cup \mathcal{F}_7 \cup \mathcal{F}_9 \cup \mathcal{F}_{79} \cup \mathcal{F}_{135}.
\end{aligned}$$

It is well-known that

$$\mathcal{I} \setminus \mathcal{B}, \quad \mathcal{B} \setminus (\mathcal{C} \cup \mathcal{E} \cup \mathcal{F}), \quad \mathcal{C}, \quad \mathcal{E}, \quad \mathcal{F}$$

constitute a partition of $\mathcal{I}$. It is known that

$$\mathcal{I} \setminus (\mathcal{C} \cup \mathcal{E} \cup \mathcal{F}) \subset \text{Dom}\, \mathcal{C},$$

from which and from Proposition 2.3, we have $\mathcal{Q} \subseteq \mathcal{C} \cup \mathcal{E} \cup \mathcal{F}$. It is known that the set $\mathcal{C} \cup \mathcal{F}$ is a subsolution, and so $\mathcal{C} \cup \mathcal{F} \subseteq \mathcal{Q}$. Let's see $\mathcal{E} \subset \mathcal{Q}$. Assume $x \in \mathcal{E}_1$. If $y \succ x$ then $y \notin \mathcal{C} \cup \mathcal{F}$ since $\mathcal{E} \cap \text{Dom}(\mathcal{C} \cup \mathcal{F}) = \emptyset$. So $y \in \mathcal{E} \cup \text{Dom}\mathcal{C}$. Suppose $y \in \text{Dom}\mathcal{C}$. Then there exists $z \in \mathcal{C}$ such that $z \succ y$, but $z \not\succ x$ since $\mathcal{E} \cap \text{Dom}\mathcal{C} = \emptyset$. Hence $x \in \mathcal{Q}$. Suppose $y \in \mathcal{E}$. Then $y \in \mathcal{E}_3 (\subset \text{Dom}\mathcal{E}_5)$. There exists $z \in \mathcal{E}_5$ such that $z \succ y$, but $z \not\succ x$ since $\mathcal{E}_1 \cap \text{Dom}(\mathcal{E}_1 \cup \mathcal{E}_5) = \emptyset$. Hence $x \in \mathcal{Q}$. So $\mathcal{E}_1 \subset \mathcal{Q}$. By permutation, we see $\mathcal{E}_3 \cup \mathcal{E}_5 \subset \mathcal{Q}$. Consequently we have that $\mathcal{Q} = \mathcal{C} \cup \mathcal{E} \cup \mathcal{F}$. Note that the set $\mathcal{C} \cup \mathcal{F}$ is a subsolution and it is the supercore[2].

[2]See Roth (1976), esp. p.48.

The next example says that the set $\mathcal{Q}$ is not always a convex set.

Example 2.2. (Lucas 1969) Let $n = 8$ and

$$v(N) = 4, v(1467) = 2, v(12) = v(34) = v(56) = v(78) = 1$$

and $v(S) = 0$ for all other S. Let

$$\mathcal{B} = \{x \in \mathcal{I} : x(12) = x(34) = x(56) = x(78) = 1\}.$$

For $i = 1, 4, 6, 7$, let

$$\mathcal{F}_i = \mathcal{B} \cap \{x \in \mathcal{I} : x_i = 1\}.$$

The core is

$$\mathcal{C} = \{x \in \mathcal{B} : x(1467) \geq 2\}.$$

It is known that

$$\mathcal{K} = \mathcal{C} \cup \mathcal{F}_1 \cup \mathcal{F}_4 \cup \mathcal{F}_6 \cup \mathcal{F}_7$$

is a unique solution which is nonconvex. Let's see $\mathcal{Q} = \mathcal{K}$. It is known that $\mathcal{I} \setminus \mathcal{B} \subseteq \text{Dom}\, \mathcal{C}$, which implies $\mathcal{Q} \subseteq \mathcal{B}$. Let $x \in \mathcal{B} \setminus \mathcal{K}$. Then $x(1467) < 2$ and $x_i < 1$ for $i = 1, 4, 6, 7$. Define $y \in \mathcal{B}$ by

$$x_i < y_i < 1, \text{ for } i = 1, 4, 6, 7, y(1467) = 2, \text{ and } y(i, i+1) = 1, \text{ for } i = 1, 2, 3, 4.$$

Then $y \succ x$ via $\{1, 4, 6, 7\}$ and $y \in \mathcal{C}$. So $x \in \text{Dom}\mathcal{C}$ and $x \notin \mathcal{Q}$. Hence $\mathcal{Q} = \mathcal{K}$.

3. A Subsolution and the Nucleolus

In this section we examine an inclusion relation between the nucleolus and the $\mathcal{Q}$.

Let v be a game. For $x \in \mathcal{I}(v)$ let $\theta(x)$ be the 2^n-vector whose components are the numbers $e(S, x), S \subseteq N$, arranged in nonincreasing order, i.e., $\theta(x)_i \geq \theta(x)_j$ whenever $1 \leq i \leq j \leq 2^n$. We say that $\theta(x)$ is lexicographically smaller than $\theta(y)$, denoted $\theta(x) <_L \theta(y)$, if and only if there is an index k such that $\theta(x)_i = \theta(y)_i$ for all $i < k$, and $\theta(x)_k < \theta(y)_k$. We write $\theta(x) \leq_L \theta(y)$ for not $\theta(y) <_L \theta(x)$. The *nucleolus* for v is the set $\mathcal{N}$ of vectors in $\mathcal{I}$ that minimizes θ in the lexicographic ordering, i.e.,

$$\mathcal{N} = \{x \in \mathcal{I} : \theta(x) \leq_L \theta(y) \text{ for all } y \in \mathcal{I}\}.$$

It is known that the nucleolus is included in the core whenever the core is non-empty. So the nucleolus is included in the set $\mathcal{Q}$ by Proposition 2.3 whenever the core is non-empty. Since the nucleolus satisfies the symmetry, Proposition 2.5 implies that the nucleolus is included in the set $\mathcal{Q}$ when the game is symmetric.

Proposition 3.1. Assume $v(S) = 0$ for S such that $|S| \leq n-2$. The nucleolus $\mathcal{N}$ is included in the set $\mathcal{Q}$.

Proof: If $\mathcal{C} \neq \emptyset$ it holds $\mathcal{N} \subseteq \mathcal{Q}$ by Proposition 2.3 since it is known that $\mathcal{N} \subseteq \mathcal{C}$. Assume that $\mathcal{C} = \emptyset$. Let $\mathcal{N} = \{x^*\}$. Without loss of generality assume

$$e(N \setminus \{1\}, x^*) \geq \ldots \geq e(N \setminus \{n\}, x^*).$$

Since $x^* \notin \mathcal{C}$ there exists $y \in \mathcal{I}$ such that $y \succ x^*$ via $N \setminus \{i\}$ for $i \in N$. Then

$$\begin{cases} e(N \setminus \{j\}, y) > e(N \setminus \{1\}, x^*), & \forall j \neq i; \\ \\ e(N \setminus \{i\}, x^*) > e(N \setminus \{i\}, y) \geq 0. \end{cases}$$

Assume $i \geq 2$. Define $z \in \mathcal{I}$ by

$$e(N \setminus \{j\}, z) = \begin{cases} e(N \setminus \{j\}, y) + \epsilon, & j \neq 1; \\ \\ e(N \setminus \{j\}, x^*) - (n-1)\epsilon, & j = 1, \end{cases}$$

so that $e(N \setminus \{1\}, x^*) - (n-1)\epsilon \geq 0$ and $e(N \setminus \{i\}, y) + \epsilon \leq e(N \setminus \{i\}, x^*)$. That is,

$$0 < \epsilon \leq \min\{\frac{e(N \setminus \{1\}, x^*)}{n-1}, e(N \setminus \{i\}, x^*) - e(N \setminus \{i\}, y)\}.$$

We have $z \succ y$ via $N \setminus \{1\}$. In order for z to dominate x^*, it must dominate only via $N \setminus \{1\}$. This is impossible because $e(N \setminus \{i\}, z) \leq e(N \setminus \{i\}, x^*)$. So $z \not\succ x^*$.

Next assume $i = 1$. Assume $e(N \setminus \{1\}, x^*) > e(N \setminus \{2\}, x^*)$. Since the nucleolus satisfies, what is called, Property I[3], we must have $x_1^* = v(1) = 0$. Then $e(N \setminus \{1\}, x^*) > e(N \setminus \{1\}, y) \geq 0$, which implies $y_1 < 0$ contradicting $y \in \mathcal{I}$. Hence we have $e(N \setminus \{1\}, x^*) = e(N \setminus \{2\}, x^*)$. Exchange $e(N \setminus \{2\}, x^*)$ with $e(N \setminus \{1\}, x^*)$. Then it reduces to the case $i = 2$. □

4. Remarks

For 3-person games, by Proposition 3.1, the nucleolus is included in the set $\mathcal{Q}$ and also the reader could see that the set $\mathcal{Q}$ coincides with the union of all stable sets.

It is interesting to examine whether the nucleolus is included in the set $\mathcal{Q}$ or not for broader classes of games.

[3] See, for example, pp.328-332 of Owen (1995).

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